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Electrohydrodynamic instability in plane layers of fluid

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The instability of a plane layer of non-conducting fluid which is in hydrostatic equilibrium between two semi-infinite conducting fluids with surface charges is discussed for both inviscid and viscous fluid models. It is shown that for both the inviscid and viscous fluid cases, the criteria for instability are the same. Consideration is given to the relevance of the results in explaining the mechanism by which the presence of an electric field promotes more readily the coalescence of water droplets on a water surface by viewing the onset of disruption of the air film as the instability of the air film under the action of the electrostatic field produced by the surface charges on the water surfaces.

1. Introduction

In a recent paper by Jayaratne & Mason (1964), a theoretical and experimental study was made of the bouncing and coalescence of small water droplets on a water surface. It was reported that coalescence of the droplets occurs more readily when electrostatic forces are present due to either a charge on the droplets or an extraneous electric field causing polarization of the droplets. Jayaratne & Mason estimated the magnitude of the electrostatic force of attraction when coalescence is induced and found that such forces are small compared with the dynamical forces acting on the droplets. It was therefore concluded that the electrostatic field promotes the coalescence of droplets indirectly by helping to disrupt the air film between the droplet and the water surface. One way in which a disruption can be viewed is by the instability of the air layer under the action of the electrostatic field produced by the surface charges on the water surfaces, and this has prompted the authors to consider the instability of such a layer of non-conducting fluid between two conducting fluids which have surface charges; in the equilibrium configuration of the system considered here, the interfacial surfaces are parallel planes.

The problem considered has a similarity to the problem of gravity waves on the surface of a plane layer of conducting fluid under the influence of an electrostatic field; this problem and a similar problem for a dielectric fluid layer have been discussed by Michael (1968) for both inviscid and viscous models. It was shown that the points of transition between stable and unstable modes of oscillation are the same in both inviscid and viscous fluids, and it may be conjectured that these are particular cases of a general result, namely that where an incompressible fluid system in hydrostatic equilibrium in a Newtonian frame of reference is unstable due to the action of electrostatic surface forces, the stability characteristics giving points of transition from stable to unstable oscillations are identical for viscous and inviscid fluids. Such a system is considered here and a comparison of viscous and inviscid oscillations made in this paper confirms the result for this case.

2. Statement of the problem

Two semi-infinite homogeneous incompressible conducting fluids are separated by a plane layer of non-conducting homogeneous incompressible fluid of depth $2h$. The outer layers are of the same fluid and are charged so that there are charges $+Q$ and $-Q$ e.s.u. per unit area on the interfacial surfaces of the upper and lower fluids respectively.

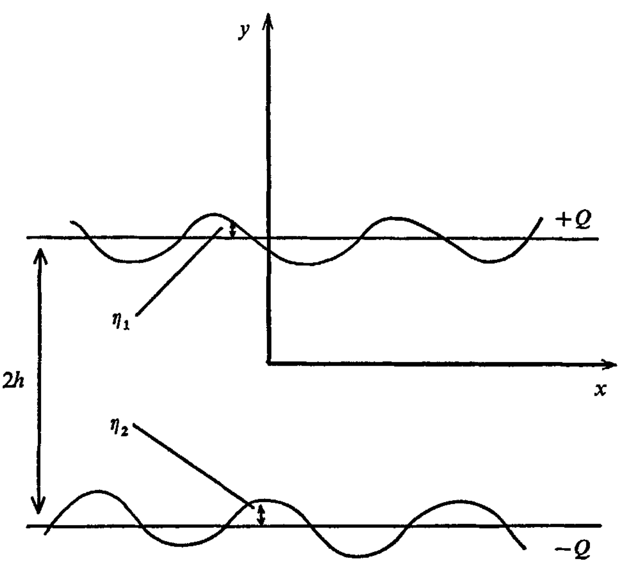

FIGURE 1

In the equilibrium state, there will be a normal stress $2\pi Q^2$ per unit area at the interfaces between the fluids due to the presence of the charges, and if Π_0 and Π_1 denote the hydrostatic pressures in the outer and inner fluid layers respectively, clearly

$$\Pi_1 = \Pi_0 + 2\pi Q^2. \tag{2.1}$$

We now consider the effect of small wave disturbances to the interfaces $y = \pm h$, given by $\zeta_1 = \eta_1 e^{i(kx-\omega t)}$ and $\zeta_2 = \eta_2 e^{i(kx-\omega t)}$ respectively, propagated in the positive x direction where x, y, z are Cartesian co-ordinates referred to axes such that $y = 0$ is the mid-plane of the inner layer and positive y is measured vertically upwards, as illustrated in figure 1.

3. The electric field

The time scale of decay of charge in the conducting fluids is $K/4\pi\sigma$, where K is the dielectric constant and σ the electrical conductivity of the fluids. If $K|\omega|/4\pi\sigma \ll 1$, this time scale is small by comparison with the time scale on which the shape of the conducting surfaces change and it follows that in such circumstances the electromagnetic field throughout the fluids may be treated as electrostatic. Consequently, within the conducting fluids, the electric field is zero and within the non-conducting fluid layer which we shall suppose has free-space dielectric constant, the electric field $\mathbf{E}$ is given by

$$\mathbf{E} = -E_0\hat{\jmath} - \nabla\chi, \tag{3.1}$$

where $E_0 = 4\pi Q$ and $\nabla^2\chi = 0$ throughout the inner layer of fluid. Thus a suitable solution for χ is of the form

$$\chi = \{F\sinh ky + G\cosh ky\}\, e^{i(kx-\omega t)}. \tag{3.2}$$

We note that for water the decay time is of the order of 10^{-4} sec. Thus for wave disturbances on a water surface, only for oscillations with periods of this order or smaller need we consider deviations from electrostatic conditions.

4. Instability of the disturbances when the fluids are inviscid

In each of the fluids, the equations governing the motion are

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho}\nabla P, \quad \operatorname{div}\mathbf{V} = 0, \tag{4.1}$$

if gravity is neglected, where $\mathbf{V}$ is the small fluid velocity and P is the perturbation pressure. The motion is irrotational, so we may satisfy the equation of continuity if $\mathbf{V} = \nabla\phi$ and ϕ is given in the three fluid layers by

$$\phi_0 = A\, e^{-ky}\, e^{i(kx-\omega t)} \quad (y \geqslant h), \tag{4.2}$$

$$\phi_1 = \{C\, e^{ky} + D\, e^{-ky}\}\, e^{i(kx-\omega t)} \quad (|y| < h), \tag{4.3}$$

$$\phi_2 = B\, e^{ky}\, e^{i(kx-\omega t)} \quad (y \leqslant -h). \tag{4.4}$$

The corresponding pressures in the fluid layers are

$$P_0 = i\omega\rho_0 A\, e^{-ky}\, e^{i(kx-\omega t)}, \tag{4.5}$$

$$P_1 = i\omega\rho_1\{C\, e^{ky} + D\, e^{-ky}\}\, e^{i(kx-\omega t)}, \tag{4.6}$$

$$P_2 = i\omega\rho_0 B\, e^{ky}\, e^{i(kx-\omega t)}, \tag{4.7}$$

where ρ_0 and ρ_1 are the densities in the outer and inner fluid layers respectively. The conditions which must be satisfied at the interfaces are:

(1) *Tangential component of* $\mathbf{E}$ *is zero,* which requires that

$$\frac{\partial\chi}{\partial x} = -E_0\frac{\partial\zeta_1}{\partial x} \quad (y = h), \quad \frac{\partial\chi}{\partial x} = -E_0\frac{\partial\zeta_2}{\partial x} \quad (y = -h), \tag{4.8}$$

which with χ given by (3.2) in turn give

$$F = -\tfrac{1}{2}E_0(\eta_1+\eta_2)\operatorname{sech} kh, \quad G = -\tfrac{1}{2}E_0(\eta_1-\eta_2)\operatorname{cosech} kh. \tag{4.9}$$

(2) *Continuity of normal component of* $\mathbf{V}$, which requires that

$$\frac{\partial\phi_0}{\partial y} = \frac{\partial\phi_1}{\partial y} = \frac{\partial\zeta_1}{\partial t} \quad (y = h), \quad \frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} = \frac{\partial\zeta_2}{\partial t} \quad (y = -h),$$

giving
$$-A\,e^{-kh} = C\,e^{kh} - D\,e^{-kh} = -i\omega\eta_1/k, \tag{4.10}$$
$$B\,e^{kh} = C\,e^{-kh} - D\,e^{kh} = -i\omega\eta_2/k. \tag{4.11}$$

(3) *Continuity of normal stress*, which requires that
$$P_0 = P_1 + T\frac{\partial^2\zeta_1}{\partial x^2} - \frac{E_0}{4\pi}\frac{\partial\chi}{\partial y} \quad (y = h),$$
$$P_2 = P_1 - T\frac{\partial^2\zeta_1}{\partial x^2} - \frac{E_0}{4\pi}\frac{\partial\chi}{\partial y} \quad (y = -h),$$
where T is the interfacial surface tension. These equations yield
$$i\omega\rho_0 A\,e^{-kh} = i\omega\rho_1\{C\,e^{kh} + D\,e^{-kh}\} - Tk^2\eta_1 - \frac{kE_0}{4\pi}\{F\cosh kh + G\sinh kh\}, \tag{4.12}$$
$$i\omega\rho_0 B\,e^{kh} = i\omega\rho_1\{C\,e^{-kh} + D\,e^{kh}\} + Tk^2\eta_2 - \frac{kE_0}{4\pi}\{F\cosh kh - G\sinh kh\}. \tag{4.13}$$

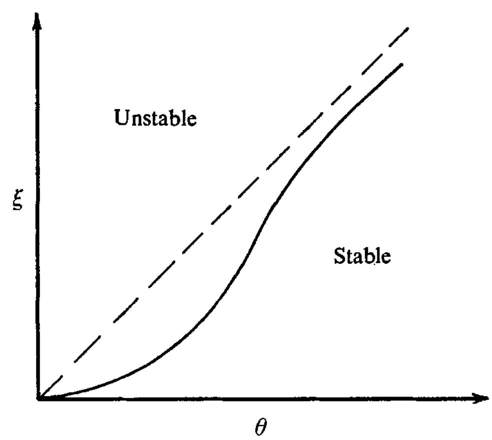

FIGURE 2

Clearly equations (4.10) and (4.11) give A, B, C and D in terms of η_1 and η_2. Substitution of these coefficients together with F and G from (4.9) into (4.12) and (4.13) gives the following relations:
$$[\omega^2k^{-1}(\rho_0 + \rho_1\tanh kh) - Tk^2 + (E_0^2k/4\pi)\tanh kh]\,(\eta_1 + \eta_2) = 0, \tag{4.14}$$
$$[\omega^2k^{-1}(\rho_0 + \rho_1\coth kh) + Tk^2 + (E_0^2k/4\pi)\coth kh]\,(\eta_1 - \eta_2) = 0. \tag{4.15}$$
We may thus consider independently the propagation of symmetric wave disturbances for which $\eta_1 + \eta_2 = 0$ and antisymmetric wave disturbances for which $\eta_1 - \eta_2 = 0$. When $\eta_1 = -\eta_2$, equation (4.15) gives the dispersion relation
$$\frac{\omega^2}{k^2} = \frac{Tk - (E_0^2/4\pi)\coth kh}{\rho_0 + \rho_1\coth kh}. \tag{4.16}$$
The denominator in the right-hand side of equation (4.16) is positive, therefore the disturbances are unstable when
$$\theta\tanh\theta < \xi = E_0^2h/4\pi T = 4\pi Q^2h/T,$$

where $\theta = kh$. Thus for a given value of ξ, disturbances are unstable for wave-numbers $k < \theta^*/h$ where $\theta^* \tanh \theta^* = \xi$ and furthermore, no matter how small the electric field E_0, sufficiently long symmetric wave disturbances are always unstable. The unstable and stable regions of the (θ, ξ) plane are indicated in figure 2.

When $\eta_1 = \eta_2$, equation (4.14) gives

$$\frac{\omega^2}{k^2} = \frac{Tk - (E_0^2/4\pi)\tanh kh}{\rho_0 + \rho_1 \tanh kh}. \tag{4.17}$$

Therefore the disturbances are unstable when $\theta \coth \theta < \xi$. Hence antisymmetric waves of sufficiently large wavelengths are stable or unstable according as ξ is less than or greater than unity. The unstable and stable regions of the (θ, ξ) plane are in this case shown in figure 3.

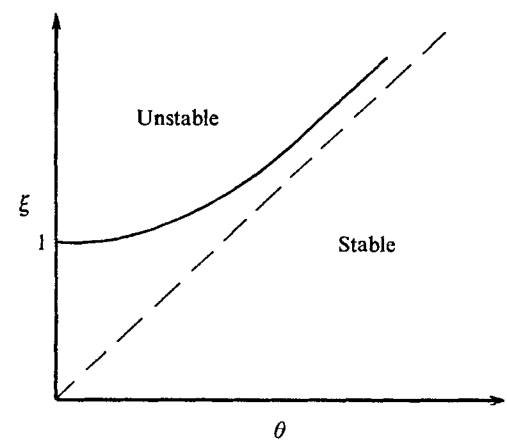

FIGURE 3

5. The instability of the disturbances when the fluids are viscous

The equations now governing the motion of any of the fluid layers are to the first order in small quantities

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}, \quad \operatorname{div} \mathbf{V} = 0, \tag{5.1}$$

where again $\mathbf{V}$ and P are the fluid velocity and perturbation pressure respectively, ν is the kinematic viscosity and gravity is neglected. The equations may be satisfied if the Cartesian components u, v of $\mathbf{V}$ are expressed in terms of a stream function Ψ defined by

$$u = \partial \Psi/\partial y, \quad v = -\partial \Psi/\partial x, \quad \Psi = \psi(y)\, e^{i(kx-\omega t)}, \tag{5.2}$$

where $$(D^2 - m^2)(D^2 - k^2)\psi = 0;$$

the operator D denotes d/dy and

$$m^2 = k^2 - i\omega/\nu. \tag{5.3}$$

The pressure is then given by $P = p(y)\,e^{i(kx-\omega t)}$, where

$$i\kappa p(y) = \rho\nu D(D^2-m^2)\,\psi. \tag{5.4}$$

The boundary conditions which must now be satisfied at the interfaces are as follows:

(1) *Tangential component of* **E** *is zero.* Equations (4.8) and (4.9) still hold for viscous fluids.

(2) *Continuity of normal component of* **V**. This requires that

$$\frac{\partial \zeta_1}{\partial t}+\frac{\partial \Psi}{\partial x} = 0 \quad (y = h), \quad \frac{\partial \zeta_2}{\partial t}+\frac{\partial \Psi}{\partial x} = 0 \quad (y = -h),$$

giving

$$\psi_0(h) = \omega k^{-1}\eta_1 = \psi_1(h), \tag{5.5}$$

$$\psi_1(-h) = \omega k^{-1}\eta_2 = \psi_2(-h), \tag{5.6}$$

where ψ_0, ψ_1, ψ_2 are the y dependent parts of the stream functions appropriate for $y > h$, $|y| < h$, $y < -h$ respectively.

(3) *Continuity of tangential component of* **V**. This is satisfied when

$$\psi_0'(h) = \psi_1'(h), \tag{5.7}$$

$$\psi_1'(-h) = \psi_2'(-h). \tag{5.8}$$

(4) *Continuity of tangential stress.* This condition is satisfied if

$$\mu_0(D^2+k^2)\psi_0(h) = \mu_1(D^2+k^2)\psi_1(h), \tag{5.9}$$

$$\mu_0(D^2+k^2)\,\psi_2(-h) = \mu_1(D^2+k^2)\,\psi_1(-h), \tag{5.10}$$

where μ_0 and μ_1 are the viscosities of the outer and inner fluid layers respectively.

(5) *Continuity of normal stress.* This condition is satisfied if

$$p_0(h)+2i\mu_0 k\psi_0'(h) = p_1(h)+2i\mu_1 k\psi_1'(h) - Tk^2\eta_1 - (E_0/4\pi)\,\chi'(h)\,e^{-i(kx-\omega t)}, \tag{5.11}$$

$$p_2(-h)+2i\mu_0 k\psi_2'(-h) = p_1(-h)+2i\mu_1 k\psi_1'(-h)+Tk^2\eta_2-(E_0/4\pi)\,\chi'(-h)\,e^{-i(kx-\omega t)}, \tag{5.12}$$

where p_0, p_1 and p_2 are pressures appropriate for $y > h$, $|y| < h$, and $y < -h$ respectively.

We may again treat the symmetric and antisymmetric parts of the disturbances independently. For symmetric wave disturbancces of the interfaces $y = \pm h$,

$$\zeta_1 = \eta_s e^{i(kx-\omega t)} = -\zeta_2. \tag{5.13}$$

Taking account of the symmetry of the fluid motions, a suitable form for the electrostatic potential χ is given by (4.9) with $G \equiv 0$, and suitable forms for the stream functions in the three fluid layers are given by

$$\psi_0 = A\,e^{-ky}+B\,e^{-m_0 y} \quad (y \geqslant h), \tag{5.14}$$

$$\psi_1 = C\sinh ky + D\sinh m_1 y \quad (|y| < h), \tag{5.15}$$

$$\psi_2 = -A\,e^{ky}-B\,e^{m_0 y} \quad (y \leqslant -h), \tag{5.16}$$

where $m_0^2 = k^2 - i\omega/\nu_0$, $m_1^2 = k^2 = k^2 - i\omega/\nu_1$. The corresponding pressures within the fluid layers will therefore be given by

$$p_0 = -\omega\rho_0 A\, e^{-ky}, \tag{5.17}$$

$$p_1 = \omega\rho_1 C \cosh ky, \tag{5.18}$$

$$p_2 = -\omega\rho_0 A\, e^{ky}. \tag{5.19}$$

Equations (5.5) to (5.10) may be easily shown to yield the following linear equations:

$$A^* + B^* = \omega k^{-1}\eta_s = C^* + D^*, \tag{5.20}$$

$$\rho_0 B^* - \rho_1 D^* = 2ik\eta_s(\mu_1 - \mu_0), \tag{5.21}$$

$$(k - m_0)\, B^* + (k \coth kh - m_1 \coth m_1 h)\, D^* = \omega\eta_s(1 + \coth kh), \tag{5.22}$$

where

$$A^* = A\, e^{-kh}, \quad B^* = B\, e^{-m_0 h}, \quad C^* = C \sinh kh, \quad D^* = D \sinh m_1 h.$$

On solving for A^*, B^*, C^* and D^* and substituting into (5.11) or (5.12), it is found that either $\eta_s = 0$ or the following dispersion relation holds:

$$\frac{\{2ik(\mu_0-\mu_1)(k-m_0)+\omega\rho_0(1+\coth kh)\} \times \{2ik(\mu_0-\mu_1)[k\coth kh - m_1\coth m_1 h] - \omega\rho_1(1+\coth kh)\}}{\{\rho_0[k\coth kh - m_1 \coth m_1 h] + \rho_1(k-m_0)\}}$$
$$+ 2ik\omega(\mu_0-\mu_1)(1-\coth kh) + \omega^2 k^{-1}(\rho_0-\rho_1) - Tk^2 + (E_0^2 k/4\pi)\coth kh = 0. \tag{5.23}$$

For antisymmetric wave disturbances of the interfaces $y = h$,

$$\zeta_1 = \eta_a\, e^{i(kx-\omega t)} = \zeta_2. \tag{5.24}$$

A dispersion relation for such wave disturbances can be similarly derived; it is found that the dispersion relation is given by (5.23) except that the hyperbolic tangent function now replaces the hyperbolic cotangent. Clearly the dispersion relations for both symmetric and antisymmetric wave disturbances are such that if $\omega = \omega_r + i\omega_i$, where ω_r and ω_i are real, satisfies either of them, then so also does $\omega = -\omega_r + i\omega_i$. We may therefore without loss of generality assume that $\mathscr{R}(\omega) \geqslant 0$.

Stability for small viscosities

On writing $n = \nu_0/\nu_1$, where n remains finite as $\nu_1 \to 0$, equation (5.3) shows that as $\nu_1 \to 0$, either $|m_0|$, $|m_1| \to \infty$ or $|m_0|$, $|m_1|$ tend to limits according as $|\nu_1 k^2/\omega|$ does or does not approach zero with ν_1. Considering first of all the situation when $|\nu_1 k^2/\omega| \to 0$ as $\nu_1 \to 0$, equation (5.3) gives

$$\left.\begin{aligned} m_1 &= \alpha\nu_1^{-\frac{1}{2}}\{1 + O(\nu_1 k^2/\omega)\},\\ m_0 &= \alpha n^{-\frac{1}{2}}\nu_1^{-\frac{1}{2}}\{1 + O(\nu_1 k^2/\omega)\},\end{aligned}\right\} \tag{5.25}$$

where $\alpha = \frac{1}{2}(1-i)(2\omega)^{\frac{1}{2}}$ and we choose the signs of m_0 and m_1 so that $\mathscr{R}(m_0)$, $\mathscr{R}(m_1) > 0$. The dispersion relation (5.23) therefore gives

$$\frac{\omega^2}{k^2}\left\{(\rho_0 + \rho_1 \coth kh) + \frac{k\rho_0\rho_1(1+\coth kh)^2}{\alpha(\rho_1 n^{-\frac{1}{2}} + \rho_0 \coth m_1 h)}\nu_1^{\frac{1}{2}} + O(\nu_1 k^2/\omega)\right\}$$
$$= Tk - (E_0^2/4\pi)\coth kh. \tag{5.26}$$

It is clear that the inviscid dispersion relation for symmetric wave disturbances, as given by (4.16) is recovered from (5.26) in the limit $\nu_1 \to 0$. On writing $\omega = \omega_0 + \epsilon$ where ω_0 denotes a value of ω when the fluids are inviscid and ϵ is the small change in ω due to the fluids having small viscosities, it may readily be shown that, provided $\omega_0 \neq 0$, ϵ is given to order $\nu_1^{\frac{1}{2}} k/\omega_0^{\frac{1}{2}}$ by

$$\epsilon = -\frac{1}{2}\left(\frac{\omega_0}{2}\right)^{\frac{1}{2}} (1+i) \frac{k\rho_0\rho_1(1+\coth kh)^2 \nu_1^{\frac{1}{2}}}{(\rho_0+\rho_1 \coth kh)(\rho_1 n^{-\frac{1}{2}} + \rho_0 \coth m_1 h)}. \tag{5.27}$$

In either of the cases $\omega_0 = i\Omega$ or $\omega_0 = \pm\Omega$ where $\Omega > 0$, it is seen that for small $\nu_1 k^2/\Omega$, $\mathscr{R}(m_1) \gg 0$. Consequently

$$\epsilon \sim -\frac{1}{2}\left(\frac{\omega_0}{2}\right)^{\frac{1}{2}} (1+i) \frac{k\rho_0\rho_1(1+\coth kh)^2 n^{\frac{1}{2}} \nu_1^{\frac{1}{2}}}{(\rho_0+\rho_1 \coth kh)(\rho_1 + n^{\frac{1}{2}}\rho_0)}. \tag{5.28}$$

Thus when $\omega_0 = i\Omega$, in which case the inviscid mode of oscillation is unstable, the effect of the fluids having small viscosities is merely to reduce the growth rate of the disturbance compared with that of the inviscid mode and when $\omega_0 = \pm\Omega$, the small viscosities cause a weak exponential damping of the disturbances compared with the stable inviscid mode of oscillation. When $\omega_0 = -i\Omega$ equation (5.27) shows that

$$\epsilon \sim -\tfrac{1}{2}\Omega^{\frac{1}{2}} \frac{k\rho_0\rho_1(1+\coth kh)^2 n^{\frac{1}{2}} \nu_1^{\frac{1}{2}}}{(\rho_0+\rho_1 \coth kh)(\rho_1 - i n^{\frac{1}{2}}\rho_0 \cot[h(\Omega/\nu_1)^{\frac{1}{2}}])}. \tag{5.29}$$

This represents a valid perturbation of order $\nu_1^{\frac{1}{2}} k/\Omega^{\frac{1}{2}}$ on the inviscid mode, but the behaviour of ϵ is in this case complicated by the term $\cot[h(\Omega/\nu_1)^{\frac{1}{2}}]$ appearing in the denominator which causes ϵ to return to zero with increasing frequency as $\nu_1 \to 0$.

We may in like manner determine the expressions for ϵ corresponding to (5.28) and (5.29) when the wave disturbances at the interfaces are antisymmetric; we find that the expressions are the same as (5.28) and (5.29) with the circular and hyperbolic cotangents replaced respectively by circular and hyperbolic tangents. It is thus clear that for antisymmetric wave disturbances, the small change ϵ in ω from a non-zero inviscid value due to the fluids having small viscosities, decays to zero with the viscosities in all cases.

To complete our analysis of the stability of the disturbances for small viscosities, we must consider the possibility of $|\nu_1 k^2/\omega| \not\to 0$ as $\nu_1 \to 0$, in which case $\omega/k^2 \to 0$ as $\nu_1 \to 0$. Equation (5.23) or its counterpart taken in the limit as $\nu_1 \to 0$ shows that this cannot occur except when

$$Tk - (E_0^2/4\pi)\coth kh = 0, \quad \text{or} \quad Tk - (E_0^2/4\pi)\tanh kh = 0,$$

respectively; that is for a point on one of the inviscid neutral stability curves shown in figures 2 and 3. The disturbances in these exceptional cases will represent the decay due to small viscosities of the steady state inviscid oscillations associated with the points on the neutral stability curves.

Stability for arbitrary viscosities

For a stable mode of oscillation to change to an unstable mode or vice versa, $\mathscr{I}(\omega) = 0$ at transition. If we suppose that at transition $\mathscr{R}(\omega) \neq 0$, we can then construct a standing wave stream function which is purely oscillatory and of the form $\Psi = \psi(x, y) \cos \Omega t$ where $\Omega > 0$. However in such a mode of oscillation, the kinetic, electrostatic and surface energies of the system are periodic in t with period $2\pi/\Omega$ which is clearly inconsistent with the action of viscosity in dissipating the energy of the system. It therefore follows that transition from stable modes to unstable modes or vice versa can only occur for viscous fluids as for inviscid fluids when $\omega = 0$. In the neighbourhood of transition ω is small, so for symmetric wave disturbances, (5.23) gives for sufficiently small $|\omega|$,

$$2ik\omega\nu_1 \left\{ (n\rho_0 - \rho_1)(1 - \coth kh) - \frac{[(\rho_1 - n\rho_0) + n\rho_0(1 + \coth kh)] \times [(\rho_1 - n\rho_0)(\coth kh - kh \operatorname{cosech}^2 kh) - \rho_1(1 + \coth kh)]}{n\rho_0[\coth kh - kh \operatorname{cosech}^2 kh] + \rho_1} \right\} + O(\omega^2/\nu_1^2 k^4) - Tk^2 + (E_0^2 k/4\pi) \coth kh = 0, \quad (5.30)$$

where $n = \nu_0/\nu_1$. Therefore to the first order,

$$\omega = -\tfrac{1}{2} i \nu_1^{-1} [Tk - (E_0^2/4\pi) \coth kh] f, \quad (5.31)$$

where $\{n\rho_0[\coth kh - kh \operatorname{cosech}^2 kh] + \rho_1\} f^{-1}$

$$= n^2\rho_0^2(\coth kh - kh \operatorname{cosech}^2 kh) + n\rho_0\rho_1(1 + \coth^2 kh) + \rho_1^2(\coth kh + kh \operatorname{cosech}^2 kh).$$

Clearly $f > 0$ so transition occurs if and only if

$$Tk = (E_0^2/4\pi) \coth kh, \quad (5.32)$$

which is the same condition at transition of symmetric wave disturbances when the fluids are inviscid. Furthermore the disturbances are stable or unstable according as

$$Tk - (E_0^2/4\pi) \coth kh$$

is positive or negative.

In a similar manner it can be shown that in the neighbourhood of transition for antisymmetric wave disturbances,

$$\omega = -\tfrac{1}{2} i \nu_1^{-1} [Tk - (E_0^2/4\pi) \tanh kh] g, \quad (5.33)$$

where $\{n\rho_0[\tanh kh + kh \operatorname{sech}^2 kh] + \rho_1\} g^{-1}$

$$= n^2\rho_0^2(\tanh kh + kh \operatorname{sech}^2 kh) + n\rho_0\rho_1(1 + \tanh^2 kh) + \rho_1^2(\tanh kh - kh \operatorname{sech}^2 kh).$$

It follows that since $g > 0$, $\omega = 0$ if and only if

$$Tk = (E_0^2/4\pi) \tanh kh, \quad (5.34)$$

which is the condition for transition for antisymmetric wave disturbances when the fluids are inviscid. The disturbances are stable or unstable according as

$$Tk - (E_0^2/4\pi) \tanh kh$$

is positive or negative.

In the discussion for the modes of oscillation when ν_0 and ν_1 are small, it was established that the oscillations grow with time when either

$$Tk - (E_0^2/4\pi)\coth kh < 0, \tag{5.35}$$

or

$$Tk - (E_0^2/4\pi)\tanh kh < 0, \tag{5.36}$$

according as the wave disturbances are symmetric or antisymmetric. Thus assuming that the time dependence of any mode is a continuous function of ν_0 and ν_1, it follows that for any values of ν_0 and ν_1, all modes are unstable when (5.35) or (5.36) hold since transition to stability cannot occur. This result may also be seen from a slightly different point of view if we consider the way in which the value of ω changes when $E_0^2h/4\pi T$ changes for fixed values of θ, ν_0 and ν_1. Here we may safely assert that when $E_0 = 0$, all waves are damped and that for sufficiently large E_0 there will be instability. If we assume that the value of ω changes continuously with E_0, we recover the above result that the stability characteristics shown in figure 2 or figure 3 apply equally in viscous and inviscid fluid models.

6. Conclusion

In conclusion we consider the growth rates of small wave disturbances in the context of the problem discussed by Jayaratne & Mason (1964). The thickness of the air film is $a/1000$, where a is the radius of the undistorted drop; the geometry of the theoretical model discussed in earlier sections would thus seem a satisfactory approximation to that of the physical model. If we take the values of the physical parameters given by Jayaratne & Mason, namely

$$E_0 = 40{,}000\,\mathrm{V/cm}, \quad T = 73\,\mathrm{dynes/cm}, \quad 2h = 1{\cdot}5\times 10^{-5}\,\mathrm{cm},$$

as representative, we find that $\xi = E_0^2h/4\pi T \sim 1{\cdot}5\times 10^{-4}$. Inequality (5.36) is never satisfied for any wave-numbers in this case, but (5.35) is satisfied for wave-numbers $k < k^*$ where $k^* \sim 1{\cdot}6\times 10^3\,\mathrm{cm}^{-1}$. The time scale on which symmetric wave disturbances with such wave-numbers grow is τ where from (4.16),

$$\tau = \frac{1}{k}\sqrt{\left(\frac{h(\rho_0\tanh kh + \rho_1)}{T(\xi - kh\tanh kh)}\right)}.$$

Using $\rho_0 = 1{\cdot}0\,\mathrm{g/cm^3}$, for water and $\rho_1 = 1{\cdot}3\times 10^{-3}\,\mathrm{g/cm^3}$ for air in addition to the other given physical parameters, τ exceeds 10^{-4} sec for wave-numbers k such that either $k < 3\,\mathrm{cm}^{-1}$ or $k^* - k < 4\,\mathrm{cm}^{-1}$. The time of contact between a droplet and the water surface before rupture of the air film occurs was found by Jayaratne & Mason to be in the range 10^{-3} sec to 10^{-1} sec; the contact time increasing as the angle of incidence at which the droplet approaches the water surface decreases. Our theoretical investigation would thus suggest that in the physical model, disruption of the air film can take place by means of the rapid growth of wave disturbances of symmetric form which would bring the water surfaces together at points where $\zeta_1 < 0$ and $\zeta_2 > 0$.

REFERENCES

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